J. Appl. Maths Mechs Vol. 55, No. 6, pp. 805–811, 1991
Printed in Great Britain.

0021–8928/91 $15.00+.00

INVESTIGATION OF THE STABILITY OF PERIODIC FLOWS OF A VISCOUS FLUID†

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(*Received* 12 *March* 1990)

An exact expression is obtained for the critical Reynolds number (R_*) for loss of stability in a wide class of one-dimensional periodic flows. An evolutionary equation is derived in the case of a small subcritically ($R-R_* \ll 1$) which describes the dynamics of the secondary vortex structure.

A FLOW which is induced by a mass force which is periodic with respect to one of the coordinates is described in the most general form by a stream function

$$\psi = f(y) \tag{0.1}$$

where f is an arbitrary smooth periodic function. The simplest case, when $f = \cos y$, was considered in all preceding papers. The question of the stability of such flows was raised by Kolmogorov in 1959. The problem was investigated in the linear and weakly non-linear approximations in [1–4]. The stability of Kolmogorov flow in a non-Newtonian fluid was considered in [5, 6] and the stability of the spatial analogues in [7–10].

The aim of this paper is to investigate the stability of a wide class (0.1) of periodic unidirectional boundary-free flows of a viscous incompressible fluid.

1. FORMULATION OF THE PROBLEM. INTRODUCTION OF SLOW VARIABLES

Consider the stability of a one-dimensional periodic flow (0.1) of a viscous incompressible fluid. We will assume that all the variables are dimensionless and take the period of the function f as being equal to 2π. Since the stream function is defined, apart from an arbitrary additive constant, we shall assume that

$$\langle f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(y)\,dy = 0 \tag{1.1}$$

As in all of the preceding papers, we shall confine ourselves to two-dimensional perturbations in the stability analysis.

In the case of small subcriticality when the Reynolds number is only slightly different from the critical value R_*, it is convenient to introduce a small parameter ε using the formula

† *Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 928–933, 1991.

$$R^{-1} = R_*^{-1}(1-\varepsilon^2) \tag{1.2}$$

Let us now deform the space-time coordinates

$$T = \varepsilon^4 t, \quad X = \varepsilon x, \quad Y = y \tag{1.3}$$

Arguments in favour of this choice of scale were formulated for the first time [11–13] as applied to the investigation of the stability of convective flows. In the case of a Kolmogorov flow it was shown [14] that the use of the scale transformation (1.2) and (1.3) enables one to obtain the results of the linear theory of stability [1, 2] together with the main results of the weakly non-linear theory comparatively simply.

In the new variables, the equation for the stream function takes the form

$$\varepsilon^4 \frac{\partial}{\partial t}(\psi_{yy} + \varepsilon^2\psi_{xx}) + \varepsilon \frac{\partial(\psi_{yy}, \psi)}{\partial(x, y)} + \varepsilon^3 \frac{\partial(\psi_{xx}, \psi)}{\partial(x, y)} = \\ = R_*^{-1}(1-\varepsilon^2)(\psi_{yyyy} + 2\varepsilon^2\psi_{yyxx} + \varepsilon^4\psi_{xxxx} - f'''') \tag{1.4}$$

The new variables T, X and Y are used in (1.4) and everywhere subsequently. For convenience, these variables are denoted by the previous letters t, x and y, respectively, and the derivative of the function f with respect to y is denoted by a prime.

Since the perturbed flow is naturally assumed to be periodic with respect to y, we integrate Eq. (1.4) with respect to the period. As a result we arrive at the integral relationship

$$\varepsilon^3 \frac{\partial}{\partial t}\langle\psi_{xx}\rangle + \frac{\partial}{\partial x}\langle\psi_y\psi_{xx}\rangle = \varepsilon(1-\varepsilon^2)R_*^{-1}\langle\psi_{xxxx}\rangle \tag{1.5}$$

which is subsequently used to calculate the critical number R_*.

2. DETERMINATION OF THE CRITICAL REYNOLDS NUMBER

Let us write the solution in the form of an asymptotic expansion

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \ldots \tag{2.1}$$

In the zeroth approximation, Eq. (1.4) yields

$$\psi_{0yyyy} = f'''' \tag{2.2}$$

so that the periodic solution of Eq. (2.2) has the form

$$\psi_0 = f(y) + \Phi_0(x, t) \tag{2.3}$$

The integral relationship (1.5) in the zeroth approximation $\langle\psi_{0y}\psi_{0xx}\rangle_x = 0$ is automatically satisfied by the solution(2.3).

To a first approximation, we get from relationships (1.4) and (2.3): $f'''\Phi_{0x} + R_*^{-1}\Phi_{1yyyy} = 0$. Consequently,

$$\psi_1 = -R_*\Phi_{0x}f_1(y) + \Phi_1(x, t) \tag{2.4}$$

The functions $f_k(y)$ are frequently encountered in (2.4) and the following formulas. These are defined as the k-fold integrals of the function f (for uniformity in the notation, it is useful to employ the understanding that $f \equiv f_0$). In other words, f_k is defined recurrently

$$f'_{k+1} = f_k(y), \quad k = 0, 1, 2, \ldots \tag{2.5}$$

In order to remove the non-uniqueness in the definition of (2.5), we require that (1.1) should be satisfied for all k, that is, $\langle f_k \rangle = 0$.

These conditions uniquely define the periodic function f_k. An explicit expression for the f_k is obtained after expansion in a Fourier series

$$f_k = \sum_{n=-\infty}^{\infty}{}' C_n (in)^{-k} e^{iny}$$

where the prime in the summation sign indicates the absence of the term with $n = 0$. Incidentally, it is more convenient to use the implicit definition (2.5) in the majority of the following calculations.

To a first approximation, the integral relationship (1.5) takes the form

$$\langle \psi_{1y} \psi_{0xx} \rangle_x + \langle \psi_{0y} \psi_{1xx} \rangle_x = R_*^{-1} \langle \psi_{0xxxx} \rangle \tag{2.6}$$

All of the integrals in (2.6) can be evaluated in an elementary manner:

$$\langle \psi_{1y} \psi_{0xx} \rangle = 0, \quad \langle \psi_{0y} \psi_{1xx} \rangle = \langle f^2 \rangle R_* \Phi_{0xxx}, \quad \langle \psi_0 \rangle = \Phi_0$$

As a result, the asymptotic solvability condition (2.6) yields the required critical Reynolds number for the loss of stability of the periodic flow (0.1)

$$R_* = \langle f^2 \rangle^{-1/2} \tag{2.7}$$

The critical number R_* has only been found analytically in certain isolated problems [15]. Result (2.7) represents the rare case when it is possible to determine R_* for an extremely wide class of flows. When $f = \cos y$, we have $R_* = \sqrt{2}$ [1]. It is also possible to express R_* in terms of the Fourier coefficients of the function

$$R_* = \Big(2 \sum_{n=1}^{\infty} |C_n|^2\Big)^{-1/2}$$

We will now show that, to a second approximation, the asymptotic condition of solvability under certain conditions leads to the earlier result (2.7) for the critical number R_*. In fact, from (1.4), (2.3) and (2.4) we find an equation for ψ_2, the periodic solution of which has the form

$$\psi_2 = -R_* \Phi_{1x} f_1(y) + R_*^2 \Phi_{0x}^2 f_2(y) + R_*^2 \Phi_{0xx} \Big(\frac{1}{2} f_1^2 - 2F\Big) + \Phi_2(x, t) \tag{2.8}$$

where we denote by $F(y)$ the periodic solution of the equation

$$F'' = f^2 - \langle f^2 \rangle \tag{2.9}$$

It is expressed in an elementary manner in terms of the Fourier coefficients of the function $F(y)$:

$$F = -\sum_{n=-\infty}^{\infty}{}' \Big(\sum_{p+q=n} C_p C_q\Big) \frac{e^{iny}}{n^2} \tag{2.10}$$

To a second approximation, the integral relationship (1.5) takes the form

$$(I_{02}+I_{11}+I_{20})_x = R_*^{-1}\langle\psi_1\rangle_{xxxx}$$
$$I_{02}=\langle\psi_{0xx}\psi_{2y}\rangle,\quad I_{11}=\langle\psi_{1xx}\psi_{1y}\rangle,\quad I_{20}=\langle\psi_{2xx}\psi_{0y}\rangle \tag{2.11}$$

By using relationships (2.3), (2.4) and (2.8), let us define

$$I_{02}=0,\quad I_{11}=0,\quad \langle\psi_1\rangle=\Phi_1$$
$$I_{20}=R_*\Phi_{1xxx}\langle f^2\rangle-3R_*^2\Phi_{0xxxx}\langle f_1f^2\rangle \tag{2.12}$$

The identities

$$\langle f'f_1\rangle=-\langle f^2\rangle,\quad \langle f'f_2\rangle=0,\quad \langle f'F\rangle=\langle f_1f^2\rangle$$

are used to calculate these integrals together with the relations $\langle f_k\rangle=0$. The above identities follow directly from the definition of the functions f_k and F on integrating by parts. For example,

$$\langle f'F\rangle=-\langle fF'\rangle=\langle f_1F''\rangle=\langle f_1f^2\rangle-\langle f_1\rangle\langle f^2\rangle=\langle f_1f^2\rangle$$

When account is taken of relationships (2.12), we get

$$(R_*^2\langle f^2\rangle-1)\,\Phi_{1xxxx}-3\langle f_1f^2\rangle R_*^3\Phi_{0xxxxx}=0 \tag{2.13}$$

from the asymptotic solvability condition (2.11).

Relationship (2.7) for the critical number R_* again follows from (2.13). However, in the general case when $\langle f_1f^2\rangle\neq 0$, this is insufficient to satisfy the condition of solvability (2.13). This fact indicates that the solution procedure which has been adopted [the form of the asymptotic expansion (2.1) and so on] is apparently invalid in this case. We shall therefore subsequently confine ourselves to arbitrary smooth periodic functions $f(y)$ which satisfy the supplementary condition

$$\langle f_1f^2\rangle=0 \tag{2.14}$$

All the following results turn out to be self-consistent when relation (2.4) is satisfied. In particular, Kolmogorov flows occur in the class of such flows.

3. DERIVATION OF THE EVOLUTIONARY EQUATION IN THE FUNCTION $\Phi_0(x,t)$

In order to obtain the evolutionary equation we turn to the third approximation. In this case, by taking account of the results for the preceding approximation, which were obtained earlier, after some fairly lengthy calculations we get the equation for the stream function

$$\begin{aligned}R_*^{-1}\psi_{3yyyy} = &-(\Phi_{0x}+\Phi_{2x})f'''+5\Phi_{0xxx}f'+R_*\Phi_{0*}\Phi_{1*}f''-\\ &-\Phi_{0x}\psi_{2yyy}+R_*\Phi_{1xx}[f_1f'''-(f')^2]+\\ &+R_*\Phi_{0x}\Phi_{0xx}(3ff'-2f_2f'''-f_1f'')+\\ &+R_*^2\Phi_{0xxx}[f_1(f')^2-f^2f'+(2F-\tfrac{1}{2}f_1^2)f''']\end{aligned} \tag{3.1}$$

This equation can be integrated. However, the final result is very long and, apart from the functions $f_k(y)$, contains numerous new functions of the type of $F(y)$. The explicit form of the solution is not required in order to obtain the evolutionary equation in $\Phi_0(x,t)$ and is therefore not given here.

To a third approximation, the integral relationship (1.5) takes the form

$$\frac{\partial}{\partial t}\langle\psi_{0xx}\rangle+\langle\psi_{0x}\psi_{3xx}\rangle_x+\langle\psi_{1y}\psi_{2xx}\rangle_x+\langle\psi_{2y}\psi_{1xx}\rangle_x+\langle\psi_{3y}\psi_{0xx}\rangle_x=$$
$$=R_*^{-1}\langle\psi_2-\psi_0\rangle_{xxxx} \tag{3.2}$$

Using (2.3), (2.4) and (2.8), we determine some of the integrals in (3.2)

$$\langle\psi_2-\psi_0\rangle=\Phi_2-\Phi_0+\frac{1}{2}R_*^2\langle f_1^2\rangle\Phi_{0xx},\quad \langle\psi_{3y}\psi_{0xx}\rangle=0$$
$$\langle\psi_{2y}\psi_{1xx}\rangle=-R_*^3\Phi_{0xxx}(\Phi_{0x}^2\langle f_1^2\rangle+2\Phi_{0xx}\langle f_2f^2\rangle)$$
$$\langle\psi_{1y}\psi_{2xx}\rangle=2R_*^3\Phi_{0*}[(\Phi_{0x}\Phi_{0xxx}+\Phi_{0xx}^2)\langle f_1^2\rangle+$$
$$+\Phi_{0xxxx}\langle f_2f^2\rangle$$

In order to evaluate these integrals, we use the supplementary identities

$$\langle fF\rangle=\langle f_2f^2\rangle,\quad \langle f_1F'\rangle=-\langle f_2f^2\rangle,\quad \langle ff_1^2\rangle=0$$

together with the identities which have been previously obtained.

When account is taken of the equality

$$\langle\psi_{0y}\psi_{3xx}\rangle=\langle f'\psi_3\rangle_{xx}$$

it remains to evaluate the integral of ψ_3f' over the period. By integrating by parts, we reduce this integral to the form

$$\langle f'\psi_3\rangle=\langle f_3\psi_{3yyyy}\rangle \tag{3.3}$$

and use Eq. (3.1) for the fourth derivative of ψ_3. As a result, we determine the integral (3.3) after some extremely long but straightforward calculations.

On collecting the results together we note the absence of terms with Φ_2 and Φ_1 from the final expression (3.2). In fact, terms of the first type constitute the expression $(R_*^2\langle f^2\rangle-1)\Phi_{2xxxx}$ which is equal to zero in view of (2.7) while terms of the second type constitute the expression $\langle f_1f^2\rangle\Phi_{1xxxxx}$ which is also equal to zero by virtue of relation (2.14).

After a double integration with respect to x, we arrive at the required evolutionary equation

$$\frac{\partial\Phi_0}{\partial t}+\frac{2}{R_*}\frac{\partial^2\Phi_0}{\partial x^2}+R_*^3\left[\alpha\frac{\partial\Phi_0}{\partial x}\frac{\partial^3\Phi_0}{\partial x^3}+\beta\left(\frac{\partial\Phi_0}{\partial x}\right)^2\frac{\partial^2\Phi_0}{\partial x^2}+\gamma\frac{\partial^4\Phi_0}{\partial x^4}\right]=0 \tag{3.4}$$
$$\alpha=4\langle f_2f^2\rangle,\quad \beta=-2\langle f_1^2\rangle$$
$$\gamma=\langle f_3f_1(f')^2\rangle-\langle f_3f^2f'\rangle+\left\langle f_3f'''\left(2F-\frac{1}{2}f_1^2\right)\right\rangle+\frac{9}{2}\langle f^2\rangle\langle f_1^2\rangle$$

Repeated integration by parts is carried out in order to reduce the moment γ to a more compact form. Here, together with the definitions (2.5) and (2.9), we shall constantly make use of the periodicity of all of the functions F, f_k ($k=0, 1, 2, \ldots$). The non-trivial components of the moment γ turn out to be:

$$\langle f_3f_1^2f'''\rangle=2\langle f_3f_1(f')^2\rangle-5\langle f^2f_1^2\rangle-\frac{8}{3}\langle f_2f^3\rangle$$
$$\langle f_3Ff'''\rangle=-\langle f^2F\rangle+\frac{3}{2}\langle f^2f_1^2\rangle-\frac{7}{3}\langle f_2f^3\rangle-\frac{9}{2}\langle f^2\rangle\langle f_1^2\rangle$$
$$\langle f_3f^2f'\rangle=-\frac{1}{3}\langle f_2f^3\rangle,\quad \langle f^2F\rangle=-\langle(F')^2\rangle$$

so that we finally obtain

$$\gamma = 2\,\langle (F')^2\rangle + {}^{11}/_2\,\langle f^2 f_1^2\rangle - 3\,\langle f_2 f^3\rangle - {}^9/_2\,\langle f^2\rangle\,\langle f_1^2\rangle \tag{3.5}$$

In the case of Kolmogorov flow, we have

$$f = \cos y,\quad f_1 = \sin y,\quad f_2 = -\cos y,\quad F = -{}^1/_8 \cos 2y$$

and hence from (3.4) and (3.5) we obtain

$$\alpha = 0,\quad \beta = -1,\quad \gamma = {}^3/_4.$$

in complete accord with the results in [14].

As a more complex example, let us consider the situation when the function f consists of two harmonics

$$f = C_1 e^{iy} + C_2 e^{2iy} + (\text{c.c}) \tag{3.6}$$

where we denote by (c.c.) the complex-conjugate terms. We determine the functions $f_k(y)$ from (3.6) and use relationship (2.10) in the case of the expression $F(y)$

$$F = -2C^*_1 C_2 e^{iy} - {}^1/_4 C_1^2 e^{2iy} - {}^1/_4 C_1 C_2 e^{3iy} - \\ - {}^1/_{16} C_2^2 e^{4iy} + (\text{c.c.}) \tag{3.7}$$

When account is taken of (3.6) and (3.7), we find the moments

$$\alpha = -18\mathrm{Re}\,(C_1^2 C_2^*),\quad \beta = -4\,|\,C_1\,|^2 - |\,C_2\,|^2 \\ \gamma = 12\,|\,C_1\,|^4 + 3\,|\,C_2\,|^4 + {}^{167}/_4\,|\,C_1\,|^2\,|\,C_2\,|^2 \tag{3.8}$$

We will also give an expression for the supplementary condition of consistency with (2.14)

$$\langle f_1 f^2\rangle = 3\,\mathrm{Im}\,(C_1^2 C_2^*) = 0 \tag{3.9}$$

If follows from (3.9) that the complex coefficients C_1 and C_2 depend on the three real parameters a, b and θ:

$$C_1 = ae^{i\theta},\quad C_2 = be^{2i\theta}$$

so that the function f, which consists of two harmonics and satisfies the consistency condition (2.14) is written in the form

$$f = 2\,(a \cos y + b \cos 2y) \tag{3.10}$$

when account is taken of the arbitrariness in the choice of the origin from where the variables are measured.

We note that $\gamma > 0$ and $\beta < 0$ in the case of the flow (3.10), as can be seen from (3.8). Only the term $\alpha\Phi_{0x}\Phi_{0xxx}$ may turn out to be zero in Eq. (3.4). It is clear from (3.8) that this is only possible when $a = 0$ or $b = 0$, that is, essentially in the case of Kolmogorov flow.

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Translated by E.L.S.